Coupling volume and surface integral formulations for eddy current problems on general meshes

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Integral formulations are attractive for solving eddy current problems in complex electromagnetic systems, since they do not require the discretisation of the complement of the conducting structures. This paper addresses the coupling of Volume Integral (VI) and Surface Integral (SI) formulations for eddy current problems on general star-shaped polyhedral and polygonal meshes.

Index Terms—**eddy currents, volume integral, surface integral, cohomology, coupling**

I. Introduction

INTEGRAL Methods (IMs) are attractive for solving eddy current problems in complex electromagnetic systems, since they do not require the discretisation of the complement of the conducting structures, which can be hard to obtain in many cases of practical interests (e.g. in magnetic confinement fusion devices, which are made of several conducting parts/components, either thick or thin, with elaborated shapes embedded in air/vacuum, see for example Fig. 1).

IMs for the solution of 3-D eddy current problems have been developed a long time ago; see for example the VI formulation on edge/face elements for eddy current problems in [1] and the SI formulation, based on a scalar potential (ψ), in [2].

The main drawback in IMs is that a dense linear system has to be assembled and its building and solution might lead to impractical memory and computational time requirements if the problem is not carefully addressed. Nonetheless, the development of effective data compression techniques (e.g. Adaptive Cross Approximation (ACA) coupled with hierarchical matrix arithmetics or Fast Multipole Method (FMM)) has revived the research on IMs [3], [4], [5] and extended their applicability to large scale systems.

This paper addresses the coupling of Volume Integral (VI) and Surface Integral (SI) formulations for eddy current problems in conducting domains, with arbitrary geometry and topology, covered by general (polygonal/polyhedral) meshes. The proposed approach works also for non-simply connected domains, provided that a suitable cohomology basis is used as described in [6] and [7]. Here, for the sake of brevity, the methodology is presented for a simple geometry with a trivial domain only. In the full paper we will present the general case.

II. Integral formulations

The domain of interest D of the eddy-current problem is partitioned into the *external* field source region D_{ext} and the conductor region $D_c = D_v \cup D_s$, where D_v is the region of massive (thick) conductors modelled with the VI formulation presented in [6] and summarised in II-A, and D_s is the region of shell (thin) conductors modelled with the SI formulation presented in [7] and summarised in II-B.

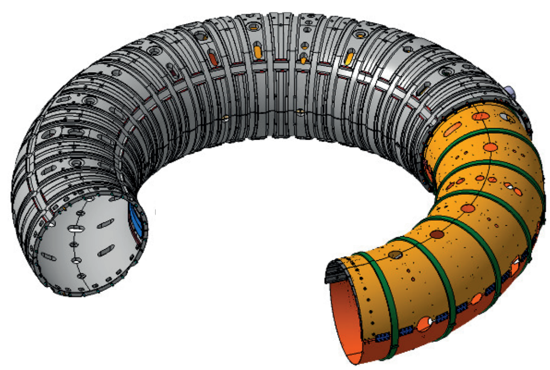

Fig. 1. Sketch of two conducting structures of RFX-mod machine: a thick stainless steel structure (outer) with several ports for diagnostics, vacuum, heating systems and a thin copper shell (inner). Courtesy of Consorzio RFX.

A. Volume Integral formulation

We cover the domain D_v with a polyhedral mesh forming the primal complex $\mathcal{K}_v$ whose oriented geometric elements are nodes n_v, edges e_v, faces f_v and volumes v_v (see Fig. 2). The interconnections of complex $\mathcal{K}_v$ are described with incidence matrix $\mathbf{G}_v$ between edges and nodes, $\mathbf{C}_c$ between faces and edges and $\mathbf{D}_v$ between elements and faces. Then, we define the array of degrees of freedom $\mathbf{T}_v$ (circulations of the electric vector potential on mesh edges) and introduce the array of the electric currents on mesh faces $\mathbf{I}_v = \mathbf{C}_v\mathbf{T}_v$. Finally, by enforcing the discrete Faraday law, through some algebra

$$[\mathbf{C}_v^T\mathbf{K}_v\mathbf{C}_v]\mathbf{T}_v = -i\omega\mathbf{C}_v^T\tilde{\mathbf{A}}_v^{ext} \tag{1}$$

where $\mathbf{K}_v = \mathbf{R}_v + i\omega\mathbf{M}_v$, $\tilde{\mathbf{A}}_v^{ext}$ are the circulations of the magnetic vector potential, due to *external* field sources, on dual edges; $\mathbf{R}_v$ and $\mathbf{M}_v$ are the resistance and inductance matrices calculated as in [6]. Note that in building (1), we cancel the rows and the columns of the linear system corresponding to boundary edges or tree edges (a standard tree-cotree gauge [1] is applied to reduce the unknowns).

B. Surface Integral formulation

Any discrete surface in D_s is covered by a mesh formed by star-shaped polygonal elements, whose oriented geometric

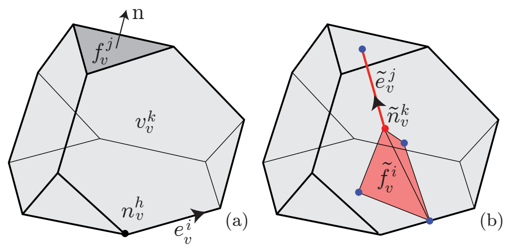

Fig. 2. Volume Integral Formulation. Geometric elements of a polyhedron $v_v^k \in \mathcal{K}_v$. a) Primal complex: An edge e_v^i and a face f_v^j. (b) Dual complex: face $\tilde{f}_v^i = D(e_v^i)$ dual to edge e_v^i ($D(\cdot)$ is the duality operator) and edge $\tilde{e}_v^j = D(f_v^j)$ dual to face f_v^j.

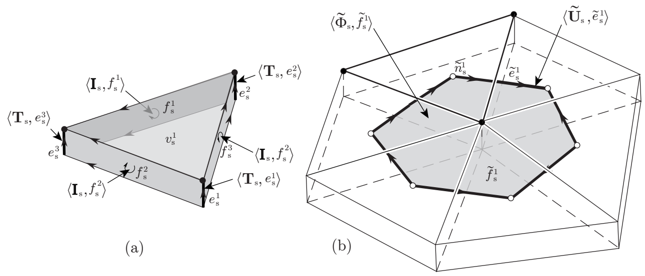

Fig. 3. Surface Integral Formulation: association of physical variables to geometric elements of the (a) primal and (b) dual complexes.

elements of $\mathcal{K}_s$ are nodes n_s, edges e_s, faces f_s and volumes v_s (see Fig. 3a). Then, the dual nodes $\tilde{n}_s$, dual edges $\tilde{e}_s$ and dual faces $\tilde{f}_s$ belonging to the dual complex $\tilde{\mathcal{K}}_s$ are constructed with the standard *barycentric subdivision*, see Fig. 3b. The interconnections of $\mathcal{K}_s$ and $\tilde{\mathcal{K}}_s$ are given in terms of the incidence matrices $\mathbf{C}_s$ between pairs (f_s, e_s) and $\tilde{\mathbf{C}}_s$ between pairs $(\tilde{f}_s, \tilde{e}_s)$, in regard to which $\tilde{\mathbf{C}}_s = \mathbf{C}_s^T$ holds, and the incidence matrix $\mathbf{D}_s$ between pairs (v_s, f_s). Then, we define the array of degrees of freedom $\mathbf{T}_s$ (circulations of the electric vector potential on mesh edges) and introduce the array of the electric currents on mesh faces $\mathbf{I}_s = \mathbf{C}_s \mathbf{T}_s$. Finally, by enforcing the discrete Faraday law, through some algebra

$$[\mathbf{C}_s^T \mathbf{K}_s \mathbf{C}_s]\mathbf{T}_s = -i\omega \mathbf{C}_s^T \tilde{\mathbf{A}}_s^{ext} \tag{2}$$

where $\mathbf{K}_s = \mathbf{R}_s + i\omega \mathbf{M}_s$, $\tilde{\mathbf{A}}_s^{ext}$ are the circulations of the magnetic vector potential, due to *external* field sources, on dual edges; $\mathbf{R}_s$ and $\mathbf{M}_s$ are the resistance and inductance matrices calculated as in [7]. Note that in building (2), we cancel the rows and the columns of the linear system corresponding to boundary edges.

C. Coupling

Then, by combining (1) and (2), the final system is

$$[\mathbf{C}^T \mathbf{K} \mathbf{C}]\mathbf{T} = -i\omega \mathbf{C}^T \tilde{\mathbf{A}}^{ext} \tag{3}$$

where $\mathbf{K} = \mathbf{R} + i\omega \mathbf{M}$, with

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_v & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_s \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} \mathbf{M}_v & \mathbf{M}_{vs} \\ \mathbf{M}_{sv} & \mathbf{M}_s \end{bmatrix} \tag{4}$$

and

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_v & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_s \end{bmatrix}, \ \mathbf{T} = \begin{bmatrix} \mathbf{T}_v \\ \mathbf{T}_s \end{bmatrix}, \ \tilde{\mathbf{A}}_v^{ext} = \begin{bmatrix} \tilde{\mathbf{A}}_v^{ext} \\ \tilde{\mathbf{A}}_s^{ext} \end{bmatrix} \tag{5}$$

$\mathbf{M}_{vs}$ and $\mathbf{M}_{sv}$ are rectangle dense matrices which couple the degrees of freedom in D_v and D_s. The computation of the entries of $\mathbf{M}$ can be performed efficiently with either *CPU* (*openMP*) or *GPU* implementations. To be able to solve realistic problems, the dense matrix on the lhs of (3) can be compressed with suitable techniques [4], [5].

III. Numerical Results

A simple test case is here considered to validate the implementation: a thin plate (thickness $\delta = 3mm$, surface resistivity $\rho_s = 5.67\mu\Omega$) is placed as a shield between a solid sphere (radius $a = 50mm$, resistivity $\rho = 0.017\mu\Omega m$) and a circular coil with rectangular cross section ($6mm \times 4mm$), fed by a sinusoidal current ($f = 50Hz$). The total number of unknowns is 4694 (4329 in D_v and 365 in D_s). The eddy currents induced in the massive and in the thin conductors are shown in Fig. 4.

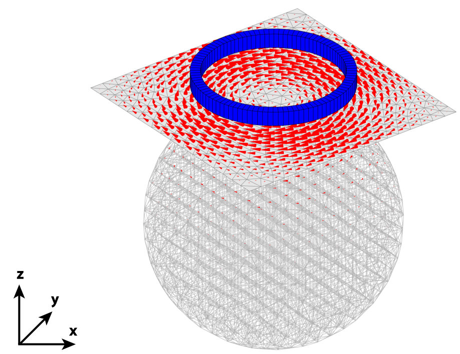

Fig. 4. Eddy currents induced in a solid sphere (discretized with 1840 polyhedra, 7256 faces, 7326 edges and 1911 nodes) and in a thin shield (discretized with 798 triangles, 1232 edges and 435 nodes) subject to the field produced by a circular coil fed by a sinusoidal current ($f = 50Hz$). Red cones: real part of the current density $\mathbf{J}$, not to scale.

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